Unit VI — Circles

Part C — Line and Segment Relationships

Lesson 1 — *Theorem 73 - "If a diameter of a circle is perpendicular to a chord of that circle, then that diameter bisects that chord."*

Objective: To investigate a special relationship between diameters and ordinary chords, as they intersect within a circle, and to prove this theorem directly, using previously accepted definitions, postulates and theorems.

Important Terms:

Chord of a Circle – A line segment whose endpoints are two points on a circle.

Diameter of a Circle – A line segment which is a chord of a circle, and passes through the center of that circle.

Midpoint of a Line Segment – A point on a line segment which is between the endpoints, and divides the given segment into two congruent segments.

Bisector of a Line Segment – Any point, line segment, ray, or line which intersects a line segment in the midpoint of the line segment, creating two congruent segments.

Arc of a Circle – Any set of continuous points on a circle.

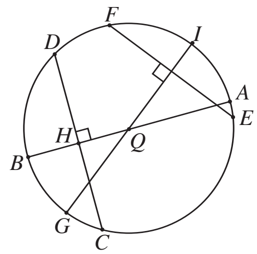
Midpoint of an Arc of a Circle – A point on an arc of a circle which is between the endpoints, and divides the given arc into two congruent arcs.

Bisector of an Arc of a Circle – Any point, line segment, ray, or line which intersects an arc in the midpoint of the arc, creating two congruent arcs.

Congruent Arcs – Arcs whose measures are equal, when in the same circle, or congruent circles.

Corollary 73a – "If a diameter of a circle is perpendicular to a chord of that circle, then that diameter bisects the arcs intercepted by that chord."

Example 1: In ⊙Q, $m\overset{\frown}{EF} = 94$, $EF = 12$, and $DH = 6$.
Determine CH, $m\overset{\frown}{FI}$, and $m\overset{\frown}{EI}$.

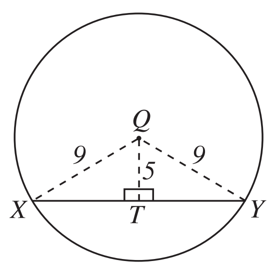


Solution: Diameter $\overline{AB} \perp \overline{DC}$. Using Theorem 73, $\overline{AB}$ bisects $\overline{DC}$. So, CH = 6

Diameter $\overline{GI} \perp \overline{EF}$. Using Corollary 73a, $\overline{GI}$ bisects $\overset{\frown}{EF}$.
So, $\overset{\frown}{FI} \cong \overset{\frown}{EI}$ and $m\overset{\frown}{FI} = m\overset{\frown}{EI} = 47$.

Example 2: Find the length of a chord that is a distance 5cm from the center of a circle with radius 9cm.

Solution: Draw ⊙Q with chord $\overline{XY}$
Draw radii $\overline{QX}$ and $\overline{QY}$. Then draw $\overline{QT} \perp \overline{XY}$ to point T on $\overline{XY}$
Use the Pythagorean Theorem.



$$(QT)^2 + (TY)^2 = (QY)^2$$
$$5^2 + (TY)^2 = 9^2$$
$$25 + (TY)^2 = 81$$
$$(TY)^2 = 56$$
$$TY = \pm\sqrt{56}$$
(TY cannot be negative)
$$TY = \sqrt{4 \cdot 14}$$
$$TY = \sqrt{4} \cdot \sqrt{14}$$
$$TY = 2\sqrt{14}$$

Since $\overline{QT} \perp \overline{XY}$ and $\overline{QT}$ is a piece of a diameter of ⊙Q, we know $\overline{QT}$ bisects $\overline{XY}$. Therefore, TX = TY.

$$\text{So, } XY = TX + TY$$
$$= 2\sqrt{14} + 2\sqrt{14}$$
$$= 4\sqrt{14} \text{ cm}$$

Lesson 1 — Exercises:

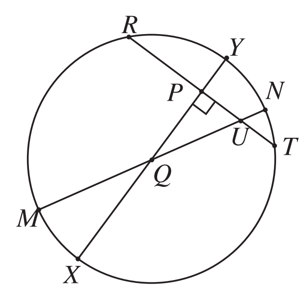
1. Prove Theorem 73 - "If a diameter of a circle is perpendicular to a chord of that circle, then that diameter bisects that chord." (Note: This is the same theorem we proved in the lesson. We are using it as an exercise to make sure you understand its proof. Use your Course Notes to check.)
   a) State the theorem.
   b) Draw and label a diagram to accurately show the conditions of the theorem.
   c) List the given information.
   d) Write the statement we wish to prove.
   e) Demonstrate the direct proof using the two column format.

2. Prove Corollary 73a - "If a diameter of a circle is perpendicular to a chord of that circle, then that diameter bisects the arcs intercepted by that chord." Use the outline given in Exercise 1.

3. Prove: The line segment joining the midpoints of the two arcs determined by a chord is the perpendicular bisector of the chord. Use the outline given in exercise 1. (This is the converse of Corollary 73a

4. State the biconditional which is the combination of exercise 3 and Corollary 73a.

In $\odot Q$, $\overline{XY} \perp \overline{RT}$. $\overline{MN}$ is a diameter of $\odot Q$. Use the diagram below for exercises 5 through 13.



5. Name a segment which is congruent to $\overline{PT}$.

6. Name the midpoint of $\overset{\frown}{RT}$.

7. Name the midpoint of $\overline{MN}$.

8. Name an arc congruent to $\overset{\frown}{TX}$.

9 Name an arc congruent to $\overset{\frown}{RY}$.

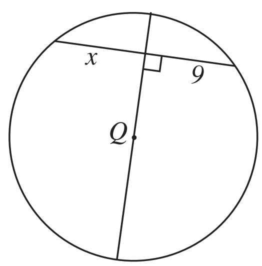
10. Name a segment congruent to $\overline{MN}$.

11. Which segment is longer, $\overline{QP}$ or $\overline{QU}$?

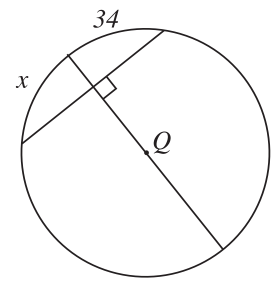
12. Which segment is longer, $\overline{XY}$ or $\overline{RT}$?

13. Answer yes or no. If point U is the midpoint of $\overline{PT}$, then point N is the midpoint of $\overset{\frown}{YT}$.

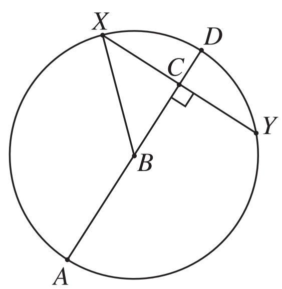
14. Find the value of *x* in ⊙Q.



15. Find the value of *x* on ⊙Q.

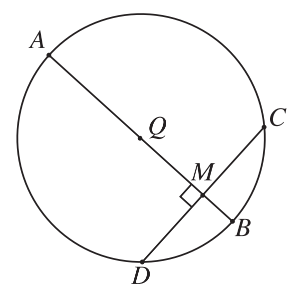


16. Find the measure of $\overline{XY}$.



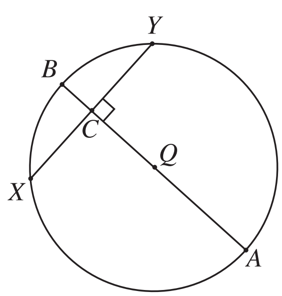
Given:
$\overline{BX} = 7$
$\overline{BC} = 4$
$\overline{AD} \perp \overline{XY}$

17. Find the measure of $\overline{QM}$.



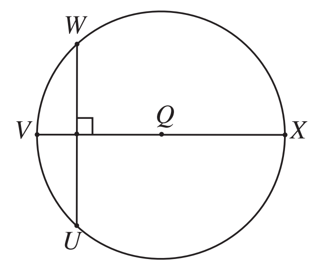
Given:
$\overline{DC} = 24$
$\overline{QA} = 18$
$\overline{AB} \perp \overline{XY}$

18. Find the measure of $\overline{QY}$.



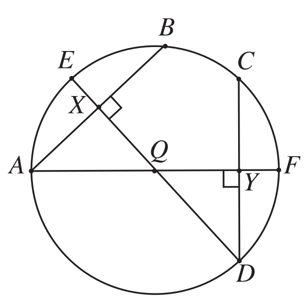
Given:
$QC = 9$
$XY = 18$
$\overline{AB} \perp \overline{XY}$

19. Find the measure of ∠VQW.



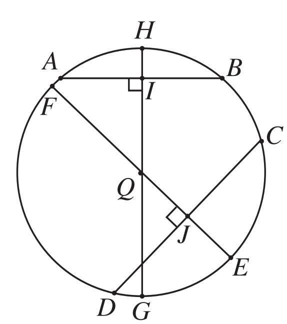
Given:
$m\overset{\frown}{UXW} = 288$
$\overline{XV} \perp \overline{UW}$

20. Find the measure of $\overline{BX}$, $\overline{DY}$, and $\overline{BF}$.



Given:
$\overline{DE} \perp \overline{AB}$
$\overline{AF} \perp \overline{CD}$
$CD = 14$
$AX = 7$
$m\overset{\frown}{BE} = 58$

21. Find the measure of $\overline{CD}$ and $\overline{AB}$.



Given:
$AB = 18$
$QI = 12$
$QJ = 10$
$m\overset{\frown}{GB} = 140$

Lesson 2 — *Theorem 74 - "If a diameter of a circle bisects a chord of the circle which is not a diameter of the circle, then that diameter is perpendicular to that chord."* *Theorem 75 - "If a chord of a circle is a perpendicular bisector of another chord of that circle, then the original chord must be a diameter of the circle."*

Objective: To investigate additional special relationships between diameters and ordinary chords, as they intersect within a circle, and to prove these theorems, using previously accepted definitions, postulates and theorems.

Important Terms:

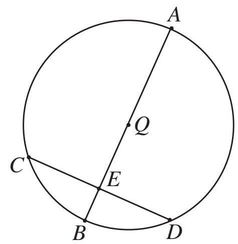
Chord of a Circle – A line segment whose endpoints are two points on a circle.

Diameter of a Circle – A line segment which is a chord of a circle, and passes through the center of that circle.

Midpoint of a Line Segment – A point on a line segment which is between the endpoints, and divides the given segment into two congruent segments.

Bisector of a Line Segment – Any point, line segment, ray, or line which intersects a line segment in the midpoint of the line segment, creating two congruent segments.

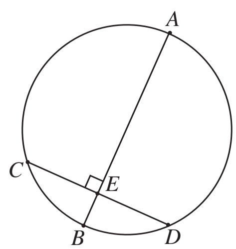
Example 1: Use the given information and figure to state a conclusion. Give a reason for your answer.



Point E is the midpoint of chord $\overline{CD}$.

Solution: $\overline{AB} \perp \overline{CD}$; Theorem 74

Example 2: Use the given information and figure to state a conclusion. Give a reason for your answer.



$\overline{AB} \perp \overline{CD}$; Point E is the midpoint of chord $\overline{CD}$.

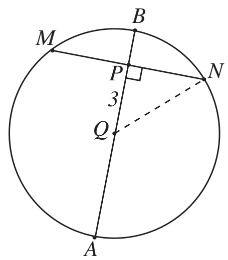
Solution: $\overline{AB}$ is a diameter of the circle; Theorem 75.

Lesson 2 — Exercises:

1. Prove Theorem 74 - "If a diameter of a circle bisects a chord of the circle which is not a diameter of the circle, then that diameter is perpendicular to that chord." (Note: This is the same theorem we proved in the lesson. We are using it as an exercise to make sure you understand its proof. Use your Course Notes to check.)
   a) State the theorem.
   b) Draw and label a diagram to accurately show the conditions of the theorem.
   c) List the given information.
   d) Write the statement we wish to prove.
   e) Demonstrate the direct proof using the two column format.

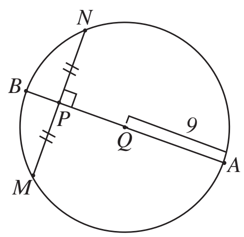
2. Prove Theorem 75 - “If a chord of a circle is a perpendicular bisector of another chord of that circle, then the original chord must be a diameter of the circle.” Use the indirect proof method.

3. Find the measure of radius $\overline{QN}$.



$\overline{AB}$ is a diameter of $\odot Q$.
$\overline{AB}$ bisects $\overline{MN}$
MN = 12

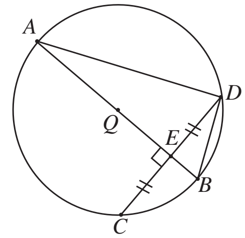
4. Find the measure of $\overline{AB}$. Explain your answer.



$\overline{AB}$ and $\overline{MN}$ are chords of $\odot Q$.
$\overline{AB}$ bisects $\overline{MN}$
$\overline{AB} \perp \overline{MN}$

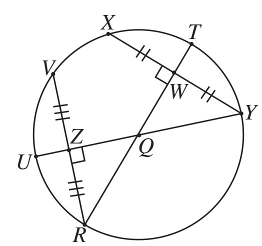
5. In the given figure to the right, $\overline{AB}$ is a diameter. $\overline{AB}$ bisects $\overline{CD}$. $m\widehat{DB} = 50$. Find:

a) $m\angle ADB$
b) $m\angle BED$
c) $m\widehat{AC}$
d) $m\angle ADC$



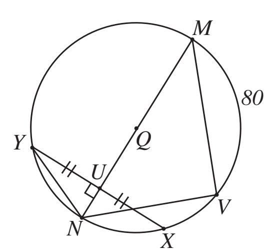
6. In $\odot Q$ at the right, $\overline{RT}$ bisects $\overline{XY}$. $\overline{RT}$ is a diameter of $\odot Q$. $\overline{UY}$ bisects $\overline{RV}$. $\overline{UY} \perp \overline{RV}$.

a) Name a segment congruent to $\overline{VZ}$.
b) Name the midpoint of $\overline{XY}$.
c) Is $\overline{RT} \perp \overline{XY}$? Justify your answer.
d) Name a segment congruent to $\overline{WY}$.
e) Is $\overline{UY}$ a diameter? Justify your answer.



7. $\overline{MN}$ and $\overline{XY}$ are chords in the given circle. $\overline{MN} \perp \overline{XY}$. $\overline{MN}$ bisects $\overline{XY}$. $m\widehat{MV} = 80$. X is the midpoint of $\widehat{NV}$. Find:

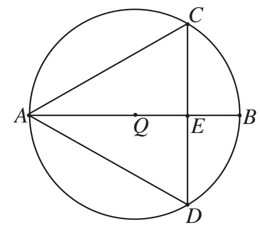
a) $m\widehat{MVN}$
b) $m\angle MVN$
c) $m\angle MNV$
d) $m\angle NYX$
e) $m\widehat{YM}$



8. Prove: If a diameter of a circle bisects one of two parallel chords (which are not diameters), it bisects the other chord.

9. Given: $\overline{AB}$ is a diameter of ⊙Q.
$\overline{CE} \cong \overline{DE}$.

Prove: $\triangle AEC \cong \triangle AED$



10. How many chords may be drawn from a point on a circle? How many diameters?

11. What is the greatest chord in a circle? An arc has how many chords?
A chord has how many arcs?

12. If a chord were extended at either or both ends, what would it become?

13. How long is the chord which is perpendicular to a tangent to a circle having a radius with a measure of 9 inches? Justify your answer.

14. Through what point does the perpendicular bisector of a chord pass?

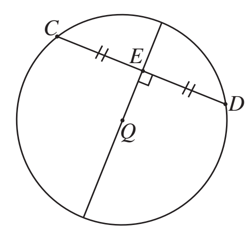
15. Prove: If a diameter of a circle bisects each of two chords which are not diameter, the chords are parallel to each other.

16. Write exercises 8 and 15 as a biconditional.

17. Prove Theorem 75 using the two column format.

Given: Chord $\overline{AB}$ bisects chord $\overline{CD}$ at point E.
$\overline{AB} \perp \overline{CD}$ at point E

Prove: $\overline{AB}$ is a diameter of ⊙Q.



18. In exercise 1, exercise 8, and exercise 15, all contain the phrase "which are not diameters". Explain why it is necessary to include this phrase in the statement of these three conditionals. Draw a diagram.

Unit VI — Circles

Part C — Line and Segment Relationships

Lesson 3 — *Theorem 76 - "If two chords intersect within a circle, then the product of the lengths of the segments of one chord, is equal to the product of the lengths of the segments of the other chord."*

Objective: To investigate the relationship between ordinary chords as they intersect with in a circle, and to prove this theorem directly, using previously accepted definitions, postulates and theorems.

Important Terms:

Chord of a Circle – A line segment whose endpoints are two points on a circle.

Means-Extremems Property of a Proportion – A property of a valid, standard proportion, which states that the product of the means is equal to the product of the extremes.

Example 1: The segments of one of two intersecting chords of a circle are 4 inches and 8 inches respectively. One segment of the other chord is 6 inches. Find the measure of the second chord.

Solution:

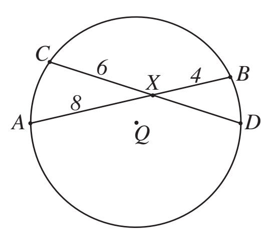
$$AX \cdot XB = CX \cdot XD$$

$$8 \cdot 4 = 6 \cdot XD$$

$$32 = 6 \cdot XD$$

$$\frac{32}{6} = XD$$

$$\frac{16 \cdot \cancel{2}}{3 \cdot \cancel{2}} = \frac{16}{3} = XD$$



The measure of the second chord is given by the expression CX + XD.

$$CD = CX + XD$$

$$CD = 6 + \frac{16}{3}$$

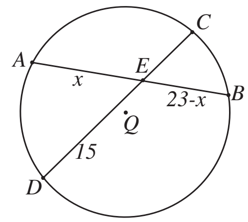
$$CD = \frac{18}{3} + \frac{16}{3}$$

$$CD = \frac{34}{3} \text{ or } 11\frac{1}{3} \text{ inches}$$

Example 2: The segments of one of two intersecting chords of a circle are 15 inches and 8 inches respectively. The length of the second chord is 23 inches. Find the lengths of the segments of the second chord.

Solution:

$$AE \cdot EB = DE \cdot EC$$
$$x(23 - x) = 15 \cdot 8$$
$$23x - x^2 = 120$$
$$0 = x^2 - 23x + 120$$
$$0 = (x - 8)(x - 15)$$
$$0 = x - 8 \text{ or } x - 15$$
$$x = 8 \text{ or } 15$$



If $x = 8$, then $23 - x = 15$.

If $x = 15$, then $23 - x = 8$.

The segments are 15 inches and 8 inches.

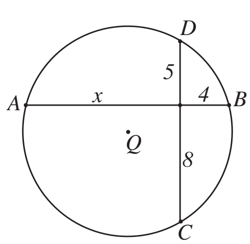
Lesson 3 — Exercises:

1. Prove Theorem 76 - "If two chords intersect within a circle, then the product of the lengths of the segments of one chord, is equal to the product of the lengths of the segments of the other chord." (Note: This is the same theorem we proved in the lesson. We are using it as an exercise to make sure you understand its proof. Use your Course Notes to check.)

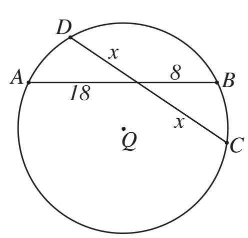
a) State the theorem.
b) Draw and label a diagram to accurately show the conditions of the theorem.
c) List the given information.
d) Write the statement we wish to prove.
e) Demonstrate the direct proof using the two column format.

In exercises 2 through 8, find the value of *x*. $\overline{AB}$ and $\overline{CD}$ are chords in ⊙Q.

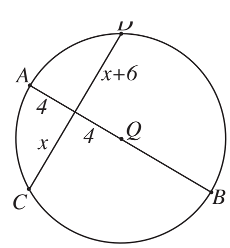
2.



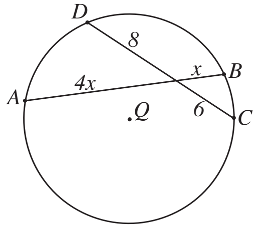
3.



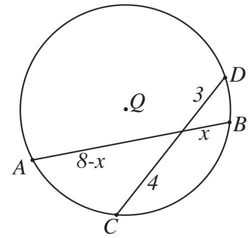
4.



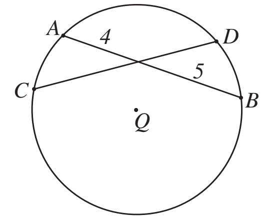
5.



6.

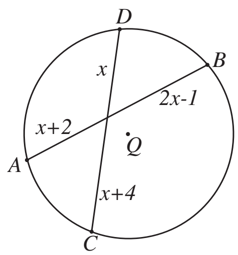


7.

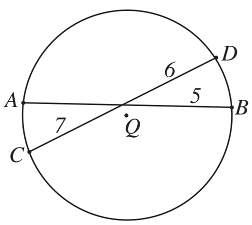


Given: CD = 12

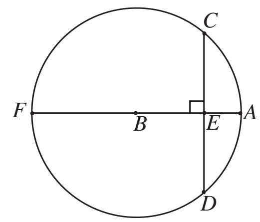
8.



9. Find the length of $\overline{AB}$.

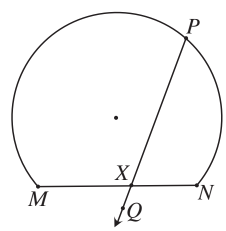


10. Find the length of $\overline{AB}$ using Theorem 76.



Given:
BE = 4
CD = 6

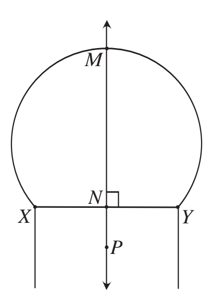
Points P, M, and N are points on a circular arc which is not a semicircle, as shown in the diagram to the right. With $\overline{MN}$ and $\overrightarrow{PQ}$.



11. What measurements must be made in order to determine whether point Q is on the circle containing $\overset{\frown}{MN}$?

12. How are these measurements used to determine if point Q is on the circle of $\overset{\frown}{MN}$?

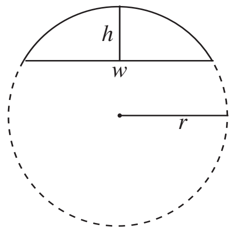
13. To find the radius of the cul-de-sac shown to the right, an architect measured $\overline{XY}$ and $\overline{MN}$ where $\overleftrightarrow{MN}$ is the perpendicular bisector of $\overline{XY}$. If XY = 44 feet and MN = 120 feet, find the radius of the cul-de-sac.



14. A formula to determine the radius of the circle containing a circular arch is given by:

$$r = \frac{\left(\frac{1}{2}w\right)^2 + h^2}{2h}$$

In this formula, "r" is the radius of the circle, "h" is the height of the arch, and "w" is the width of the arch.



Find r if w = 200 and h = 40.

Unit VI — Circles

Part C — Line and Segment Relationships

Lesson 4 — *Theorem 77 - "If two secant segments are drawn to a circle from a single point outside the circle, the product of the lengths of one secant segment and its external segment, is equal to the product of the lengths of the other secant segment and its external segment." Theorem 78 - "If a secant segment and a tangent segment are drawn to a circle, from a single point outside the circle, then the length of that tangent segment is the mean proportional between the length of the secant segment, and the length of its external segment."*

Objective: To investigate the relationship between intersecting secant segments and tangent segments of a circle, as they intersect outside that circle, and to prove these theorems directly, using previously accepted definitions, postulates and theorems.

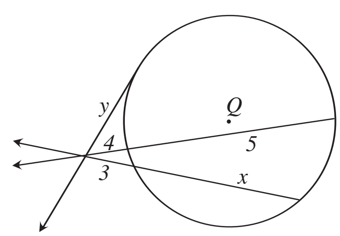
Important Terms:

Secant Line of a Circle – From the Latin word, “secare”, meaning, “to cut”, a secant of a circle is a line which intersects the circle in two distinct points. Note: We sometimes refer to a line segment as a secant segment, if that segment intersects the circle in two distinct points, and at least one of its endpoints is a point on the circle.

Tangent Line of a Circle – From the Latin word, “tangere”, meaning, “to touch”, a tangent of a circle is a line which intersects the circle in exactly one point, called the “point of tangency”, or the “point of contact”. Note: We sometimes refer to a line segment as a tangent segment, if that segment intersects the circle in such a way that the point of tangency is one of its endpoints.

Means-Extremes Property of a Proportion – A property of a valid, standard proportion, which states that the product of the means is equal to the product of the extremes.

Example 1: Find the value of *x* and *y.*



Solution:

$4(4+5) = 3(3+x)$

$4 \cdot 9 = 9 + 3x$

$36 = 9 + 3x$

$27 = 3x$

$9 = x$ Theorem 77

$\frac{4}{y} = \frac{y}{4+5}$

$4(4+5) = y^2$

$4(9) = y^2$

$36 = y^2$ Theorem 78

$0 = y^2 - 36$

$0 = (y+6)(y-6)$

$0 = y+6$ or $0 = y-6$

$-6 = y$ or $6 = y$

(y cannot be negative)

$6 = y$

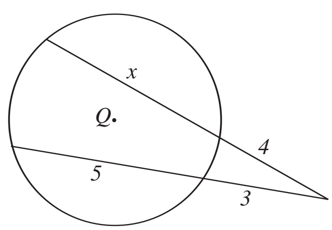
Lesson 4 — Exercises:

1. Prove Theorem 77 - "If two secant segments are drawn to a circle from a single point outside the circle, the product of the lengths of one secant segment and its external segment, is equal to the product of the lengths of the other secant segment and its external segment." (Note: This is the same theorem we proved in the lesson. We are using it as an exercise to make sure you understand its proof. Use your Course Notes to check.)
   a) State the theorem.
   b) Draw and label a diagram to accurately show the conditions of the theorem.
   c) List the given information.
   d) Write the statement we wish to prove.
   e) Demonstrate the direct proof using the two column format.

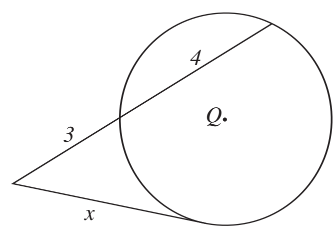
2. Prove Theorem 78 - "If a secant segment and a tangent segment are drawn to a circle, from a single point outside the circle, then the length of that tangent segment is the mean proportional between the length of the secant segment, and the length of its external segment." Use the outline given in Exercise 1.

In exercises 3 through 14, find the value of *x*.

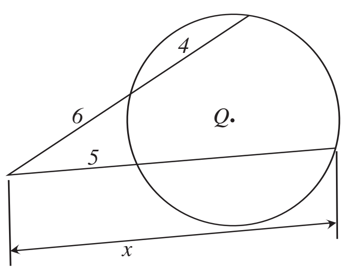
3.



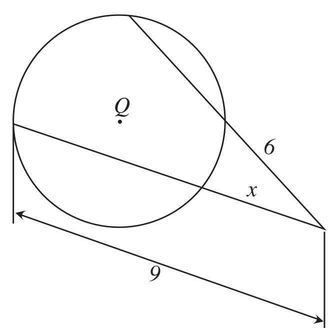
4.



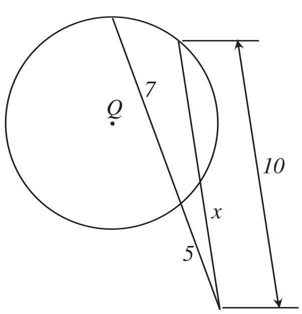
5.



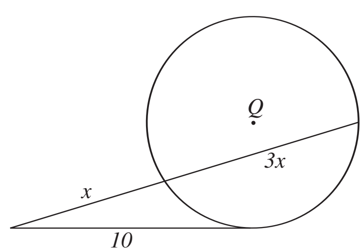
6.



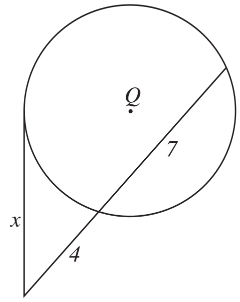
7.



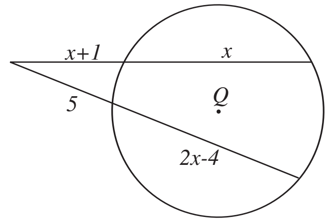
8.



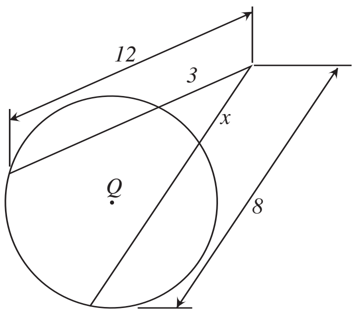
9.



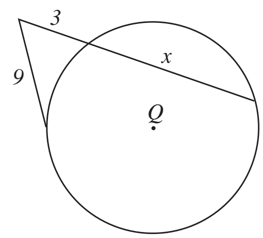
10.



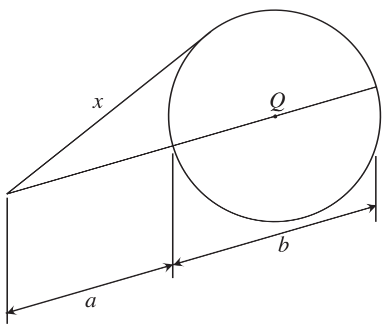
11.



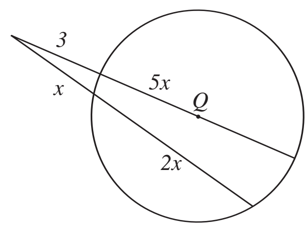
12.



13.

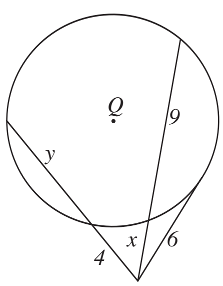


14.

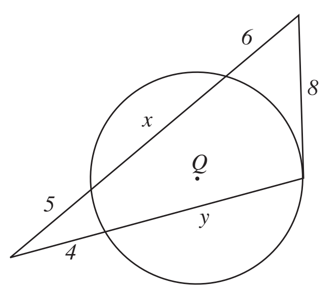


(Hint: Remember the quadratic formula)

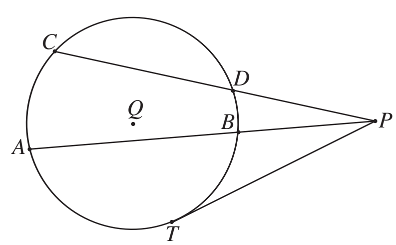
15. Find x and y.



16. Find x and y.



17. $\overline{PT}$ is tangent to ⊙Q. Find the lengths indicated.



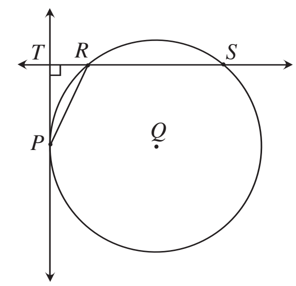
a) PT = 6; PB = 3; AB = ___________.

b) PT = 12; CD = 18; PC = ___________.

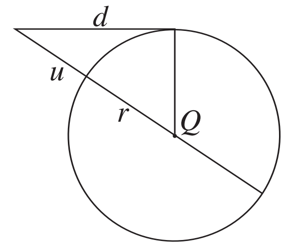
c) PD = 5; CD = 7; AB = 11; PB = ___________.

d) PB = 5; AB = 5; PD = 4; PT = __________; PC = ___________.

18. $\overleftrightarrow{PT}$ is tangent to ⊙Q. Secant $\overleftrightarrow{RS}$ is perpendicular to $\overleftrightarrow{PT}$ at point T. If RP = 8 and RT = 3, find:
a) RS
b) the distance from point Q to $\overleftrightarrow{RS}$
c) The radius of ⊙Q.



19. A secant, a radius, and a tangent of ⊙Q are shown.
a) Explain why $(r + u)^2 = r^2 + d^2$.
b) Simplify the equation in part "a" and show $d^2 = u(2r + u)$.
c) This is a special case of what Theorem?



Unit VI — Circles

Part C — Line and Segment Relationships

Lesson 5 — *Theorem 79 - "If a line is perpendicular to a diameter of a circle at one of its endpoints, then the line must be tangent to the circle, at that endpoint."*

Objective: To investigate another relationship between diameters and tangent lines of a circle, and to prove this theorem indirectly, using previously accepted definitions, postulates and theorems.

Important Terms:

Diameter of a Circle – A line segment which is a chord of a circle, and passes through the center of that circle.

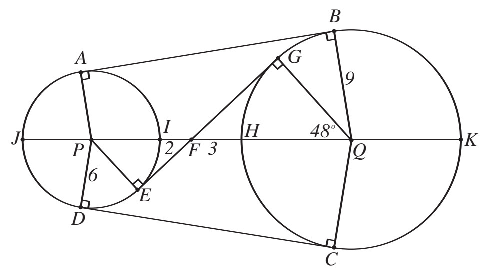
Tangent Line of a Circle – From the Latin word, "tangere", meaning, "to touch", a tangent of a circle is a line which intersects the circle in exactly one point, called the "point of tangency", or the "point of contact". Note: We sometimes refer to a line segment as a tangent segment, if that segment intersects the circle in such a way that the point of tangency is one of its endpoints.

Lesson 5 — Exercises:

1. Prove Theorem 79 using the two column format.

2. Prove Theorem 79 using the indirect proof method.

Use the figure to answer questions 3 through 15.



$\overline{PA} \perp \overline{AB}$; $\overline{PD} \perp \overline{DC}$; $\overline{PE} \perp \overline{EG}$;
$\overline{QB} \perp \overline{BA}$; $\overline{QC} \perp \overline{CD}$; $\overline{QG} \perp \overline{GE}$
QB = 9; PD = 6; FI = 2; FH = 3

3. m∠ABQ = __________.

4. $\overline{FG}$ is a __________ segment. State the Theorem which justifies this statement.

5. PA = __________.

6. m∠GFQ = __________.

7. FG = __________.

8. JK = __________.

9. m∠FPE = __________.

10. $\overline{FE}$ is a __________ segment. State the Theorem which justifies this statement.

11. FE = __________.

12. DC = __________.

13. m∠PFE = __________.

14. PF = __________.

15. $\overline{DC}$ is a tangent segment to circle __________ and circle __________.
State the Theorem which justifies this statement.

16. Theorem 79 is the converse of Corollary 68a. State the relationship as a biconditional.

Unit VI — Circles

Part C — Line and Segment Relationships

Lesson 6 — *Theorem 80 - "If two tangent segments are drawn to a circle from the same point outside the circle, then those tangent segments are congruent."*

Objective: To investigate another relationship between diameters and tangent lines of a circle, and to prove this theorem indirectly, using previously accepted definitions, postulates and theorems.

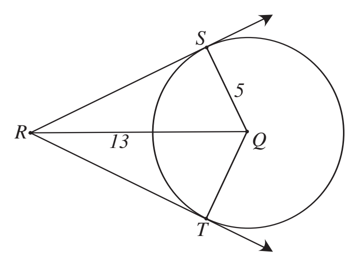
Important Terms:

Diameter of a Circle – A line segment which is a chord of a circle, and passes through the center of that circle.

Tangent Line of a Circle – From the Latin word, "tangere", meaning, "to touch", a tangent of a circle is a line which intersects the circle in exactly one point, called the "point of tangency", or the "point of contact". Note: We sometimes refer to a line segment as a tangent segment, if that segment intersects the circle in such a way that the point of tangency is one of its endpoints.

Corollary 80a – "If two tangent segments are drawn to a circle from the same point outside the circle, then the line containing that point and the center of the circle, bisects the angle formed by the two tangent segments."

Example 1: $\overline{RT}$ and $\overline{RS}$ are tangent segments to $\odot Q$. SQ = 5 and RQ = 13. Find RT and RS



Solution:

$$(RQ)^2 = (RS)^2 + (SQ)^2$$

$$(13)^2 = (RS)^2 + (5)^2$$

$$169 = (RS)^2 + 25$$

$$144 = RS^2$$

$$0 = (RS)^2 - 144$$

$$0 = (RS - 12)(RS + 12)$$

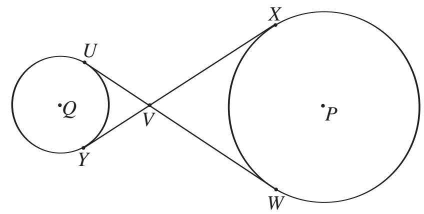
$RS = 12$ or $RS = -12$ (RS cannot be negative)

By theorem 80, $RS = RT = 12$.

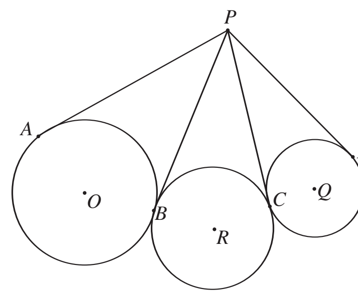
Lesson 6 — Exercises:

1. Prove Theorem 80 - “If two tangent segments are drawn to a circle from the same point outside the circle, then those tangent segments are congruent.”

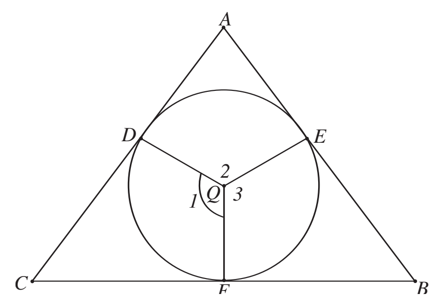
2. $\overline{UW}$ and $\overline{XY}$ are common internal tangents to ⊙Q and ⊙P. If UV = 3 and XV = 5, find UW and XY.



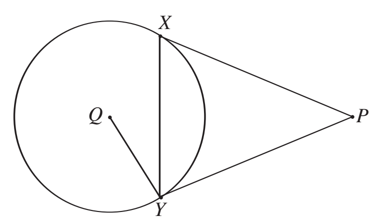
3. The diagram at the right shows tangent segments $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$ drawn to ⊙O, ⊙R,and ⊙Q. Find PD if PA = 12



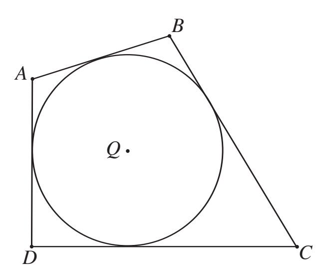
4. Tangent segments are drawn from points A, B, and C to ⊙Q so as to form △ABC about ⊙Q. If ∠1 ≅ ∠2 ≅ ∠3, prove that ABC is equilateral. Use the two column format or a paragraph argument.



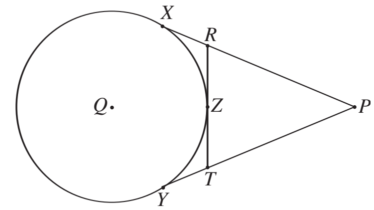
5. The diagram at the right shows tangent segments $\overline{PX}$ and $\overline{PY}$ to ⊙Q. m∠XYQ is 10 degrees. Find m∠P.



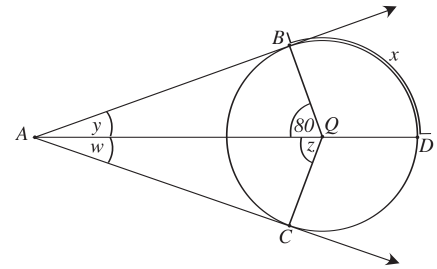
6. Quadrilateral ABCD is circumscribed about ⊙Q. Show that AB + DC = AD + BC.



7. $\overline{PX}$, $\overline{PY}$, and $\overline{RT}$ are tangents to $\odot Q$.
Show that PX + PY = PR + RT + TP.

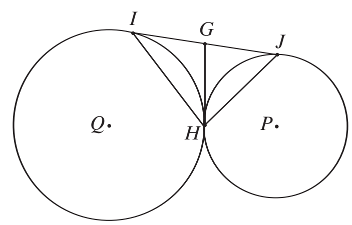


8. Determine the values of *w*, *x*, *y*, and *z*.
$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangent to $\odot Q$.



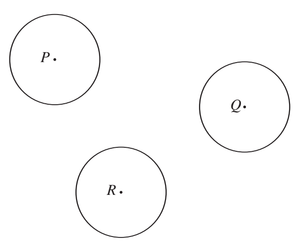
For exercises 9 and 10, refer to the following figure and given information.

Given: $\overline{IJ}$ is a common external tangent of $\odot Q$ and $\odot P$.
$\overline{GH}$ is a common internal tangent of $\odot Q$ and $\odot P$.

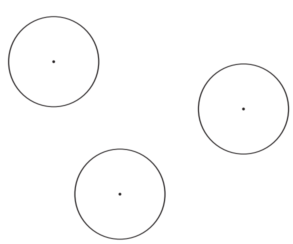


9. Prove: Point G is the midpoint of $\overline{IJ}$.
10. Prove: $\angle IHJ$ is a right angle.

11. Prove: Corollary 80a
12. How many tangents can be drawn to a circle through a point outside a circle?
13. How many tangents can be drawn to a circle through a point inside a circle?
14. How many tangents can be drawn to a circle through a point on a circle?
15. Can a diameter of a circle be a tangent of a circle?
16. Three circles are shown below. How many circles tangent to all three of the circles can be drawn?



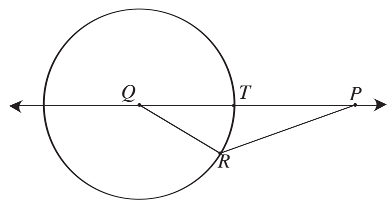
17. Three circles are shown below. Suppose the three circles represent three spheres.
a) How many planes tangent to all three spheres can exist?
b) How many spheres tangent to all three spheres can exist?



18. In space, what three-dimensional figure will be formed by all tangents to a sphere from a point outside the sphere?

19. Consider a sphere in space with all possible points of tangency on a great circle of the sphere. What three-dimensional figure will be formed by the tangents to the sphere at the points of tangency on this great circle? (A great circle is any circle drawn on the surface of a sphere whose center is at the center of the sphere.)

20. The distance from a point to a circle is measured along the line that contains the point and the center of the circle. In the figure below, the distance from point P to circle Q is TP, where points Q, T, and P are collinear. Show that if any other point R of the circle is chosen, then RP > TP.



Unit VI — Circles

Part C — Line and Segment Relationships

Lesson 7 — *Theorem 81 - "If two chords of a circle, or congruent circles are congruent, then their intercepted minor arcs are congruent." Theorem 82 - "If two minor arcs of a circle or congruent circles are congruent, then the chords which intercept them are congruent."*

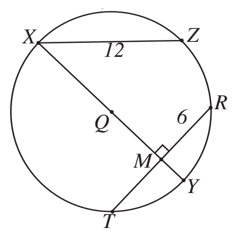
Objective: To investigate a relationship between chords and arcs of a circle, and to prove these theorems directly, using previously accepted definitions, postulates and theorems.

Important Terms:

Chord of a Circle – A line segment whose endpoints are two points on a circle.

Arc of a Circle – Any set of continuous points on a circle.

Example 1: Using the circle shown below, find TM and $m\overset{\frown}{TY}$. It is given that $\overline{XY} \perp \overline{RT}$, XZ = 12, RM = 6, and $m\overset{\frown}{XZ} = 92$



Solution: Diameter $\overline{XY}$ bisects chord $\overline{RT}$ at M. (Theorem 73)
Since RM = 6, then TM = 6 and RT = 12.
Because $\overline{XZ} \cong \overline{RT}$ and $\overset{\frown}{XZ} \cong \overset{\frown}{RT}$, then $m\overset{\frown}{RT} = 92$
$\overline{XY}$ bisects $\overset{\frown}{RT}$ (Corollary 73a)
Therefore, $m\overset{\frown}{TY} = 46$.

Lesson 7 — Exercises:

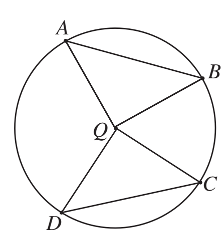
1. Prove Theorem 81 - "If two chords of a circle are congruent, then their intercepted minor arcs are congruent." (Note: This is the same theorem we proved in the lesson. We are using it as an exercise to make sure you understand its proof. Use your Course Notes to check.)
   a) State the theorem.
   b) Draw and label a diagram to accurately show the conditions of the theorem.
   c) List the given information.
   d) Write the statement we wish to prove.
   e) Demonstrate the direct proof using the two column format.

2. Using the outline given in exercise 1, prove Theorem 82.

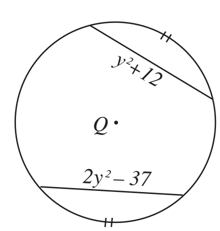
3. Prove: In the same circle or congruent circles, two chords are congruent if and only if the two chords are equidistant from the center of the circle.

4. Given: $\overline{AB}$ and $\overline{CD}$ are chords of ⊙Q.
   $\overline{AB} \cong \overline{CD}$

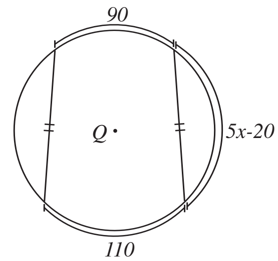
   Prove: $\angle AQB \cong \angle CQD$



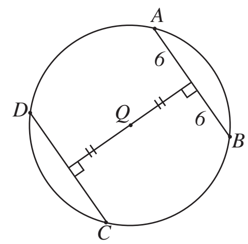
5. Find the positive value of *y* in ⊙Q.



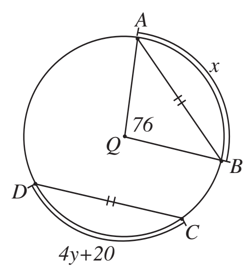
6. Find the value *x* in $\odot Q$.



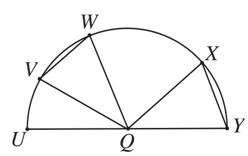
7. Find the measure of $\overline{CD}$ in $\odot Q$.



8. In $\odot Q$, $m\overset{\frown}{AB} = x$, and $\overline{AB} \cong \overline{CD}$. Find the value of *x* and *y*.

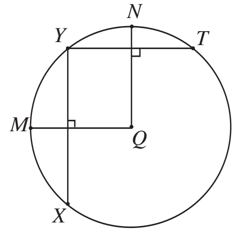


9. In semicircle Q at the right, $\overline{QV}$ bisects $\angle UQW$ and $\angle UQV \cong \angle YQX$. Explain why $\overline{VW} \cong \overline{XY}$.

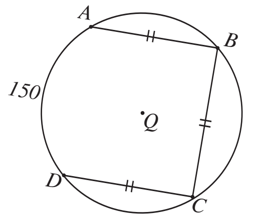


10. Six points on a circle divide the circle into six congruent arcs. Explain why the chords of these arcs form an equilateral hexagon.

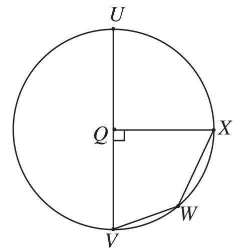
11. In $\odot Q$ at the right, $\overline{XY}$ and $\overline{YT}$ are congruent chords, $\overline{QM} \perp \overline{XY}$ and $\overline{QN} \perp \overline{YT}$. Prove that point Y is the midpoint of $\overset{\frown}{MN}$.



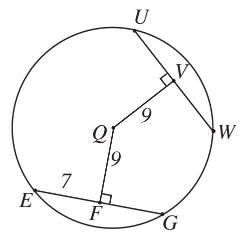
12. $m\widehat{AD} = 150$
Find $m\widehat{BC}$.



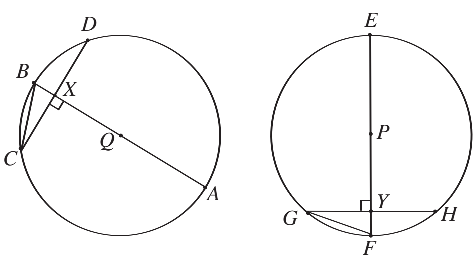
13. $\overline{XQ} \perp \overline{UV}$
$\overline{WX} \cong \overline{VW}$
Find $m\widehat{VW}$.



14. $QV = 9$
$QF = 9$
$EF = 7$
$\overline{QV} \perp \overline{UW}$
$\overline{QF} \perp \overline{EG}$
Find UW



Use the following information in exercises 15 through 17.



Given: $\overline{AB}$ is a diameter of $\odot Q$
$\overline{EF}$ is a diameter of $\odot P$.
$\overline{AB} \perp \overline{CD}$
$\overline{EF} \perp \overline{GH}$

15. Given: $\odot Q \cong \odot P$
$\overline{CD} \cong \overline{GH}$

Prove: $\widehat{CD} \cong \widehat{GH}$

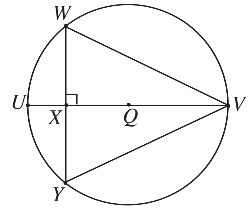
16. Given: $\odot Q \cong \odot P$
$\overline{QX} \cong \overline{PY}$

Prove: $\overline{CD} \cong \overline{GH}$

17. Given: $\overline{QX} \cong \overline{PY}$
$\odot Q \cong \odot P$

Prove: $\widehat{CD} \cong \widehat{GF}$

Use the diagram at the right for exercises 18 and 19.



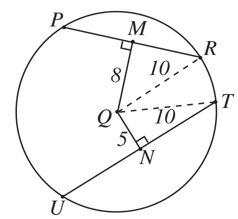
18. Given: $\overline{UV}$ is a diameter of ⊙Q
$\overline{UV} \perp \overline{WY}$

Prove: $\widehat{WV} \cong \widehat{YV}$

19. Given: $\overline{UV}$ is a diameter of ⊙Q
$\overline{UV} \perp \overline{WY}$

Prove: $\triangle VXW \cong \triangle VXY$

20. ⊙Q has a radius of 10.
Chord $\overline{UT}$ is 5 units from point Q.
Chord $\overline{PR}$ is 8 units from point Q.



a) Find PR and UT
b) Compare the distance that $\overline{PR}$ and $\overline{UT}$ are from the center, point Q.
c) Write a possible conclusion based on your observations.

21. Determine which statements are true and which are false.

a) All chords of a circle are shorter than a given diameter of the circle.

b) All chords of a circle have exactly two points in common with the circle.

c) Some chords of a circle are congruent to a given radius of the circle.

d) Chords of unequal length in a circle cannot intersect.

e) Chords of a circle congruent to a given radius have congruent minor arcs.

f) If an arc's measure is doubled, then the length of its chord is doubled.

g) If an arc's measure is doubled, then the measure of its central angle is doubled.

h) In a given circle, if one chord is twice as long as a second chord, then the shorter chord is twice as far from the center as the longer chord.

i) Congruent chords of different circles intercept congruent arcs.

j) If two chords are perpendicular, one chord is a diameter.

Unit VI — Circles

Part D — Circles and Concurrency

Lesson 1 — *Theorem 83 - "If you have a triangle, then that triangle is cyclic."*

Objective: To investigate a relationship between chords and arcs of a circle, and to prove these theorems directly, using previously accepted definitions, postulates and theorems.

Important Terms:

Concurrent – From two Latin words meaning "to run together", this term deals primarily with intersecting lines, or line segments. Formally, if two or more lines contain the same point, they are said to be concurrent. (Note: In Geometry, because it is more significant when three or more lines contain the same point, the formal geometric definition of concurrency relates to three or more lines.)

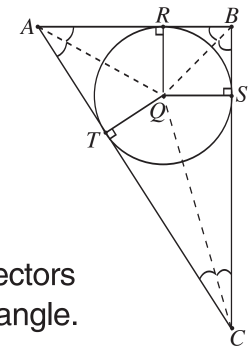
Concyclic – From two Latin words meaning "together on a circle", this term deals primarily with points on a circle. Formally, if two or more points lie on the same circle, they are said to be concyclic. (Note: In Geometry, because it is more significant when three of more points lie on the same circle, the formal geometric definition of concyclic points relates to three or more points.)

Cyclic – A specific term which refers to a relationship between polygons and circles. Formally, when there is a circle which contains all of the vertices of a polygon, then that polygon is cyclic.

Corollary 83a – "If you have a triangle, then the perpendicular bisectors of the sides are concurrent."

Corollary 83b – "If you have a triangle, then the bisectors of the angles are concurrent."

Example 1: The angle bisectors of $\triangle$ABC meet at point Q from Corollary 83b. Given that AT = 15 and AQ = 17.



a) Which segments are congruent?
b) Find QT and QS.

Solution:

a) $\overline{QT}$, $\overline{QS}$ and $\overline{QR}$ are congruent since the angle bisectors meet at a point equidistant from the sides of the triangle. Using Theorem 80, we can conclude that:
$\overline{AR} \cong \overline{AT}$, $\overline{BR} \cong \overline{BS}$, and $\overline{CS} \cong \overline{CT}$.

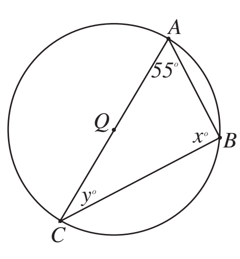
b) $\overline{QT} \perp \overline{AT}$ by corollary 68a.

$$(AT)^2 + (QT)^2 = (AQ)^2$$
$$(15)^2 + (QT)^2 = (17)^2$$
$$225 + (QT)^2 = 289$$
$$(QT)^2 = 64$$
$$QT = \pm\sqrt{64} \text{ (QT cannot be negative)}$$
$$QT = 8$$
$$QT = QS$$
$$QS = 8$$

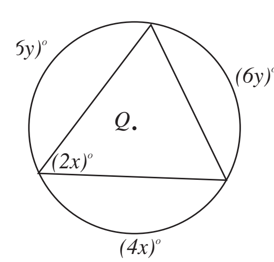
Lesson 1 — Exercises:

1. Prove Theorem 83 - “If you have a triangle, then that triangle is cyclic.”

2. Find the values of *x* and *y*.



3. Find the values of *x* and *y*.



4. Write a two column proof for Corollary 83a - “If you have a triangle, then the perpendicular bisectors of the sides are concurrent.”

5. Write a two column proof for Corollary 83b - “If you have a triangle, then the bisectors of the angles are concurrent.”

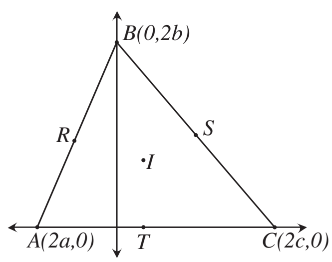
6. On a piece of paper, draw any triangle and cut out the triangle with scissors. Fold the triangle at a vertex in such a way that one side of the angle is superimposed on the other side. Repeat this for each vertex.

a) The three creases appear to ________________ the angles of the triangle.
b) Do the three creases seem to be models of concurrent lines?

7. On a piece of paper, draw any triangle and cut out the triangle with scissors. Fold the triangle in such a way that a vertex is superimposed on another vertex. Repeat this with each pair of vertices.

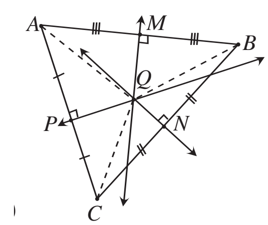
a) The three creases appear to be ________________ of the sides of the triangle.
b) Do the three creases seem to be models of concurrent lines?

8. Prove Corollary 83a using coordinates. Select a coordinate system for any triangle so that its vertices are A (2a,0), B (0, 2b) and C (2c, 0) as in the given figure.

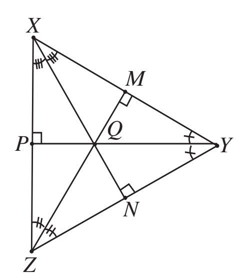


a) Find the midpoints of the sides
b) Find the equations of the lines which are the perpendicular bisectors of the sides of △ABC.
c) Solve each of the three pairs of equations to find the common intersection point.
d) Use the distance formula to show IA = IB = IC.

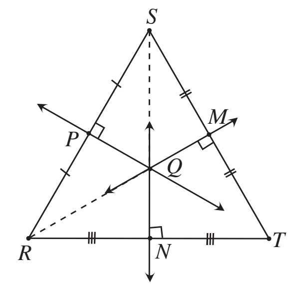
9. The perpendicular bisectors of the sides of △ABC meet at point Q. BC = 48 and QN = 7. Find QA. Give a reason for your answer.



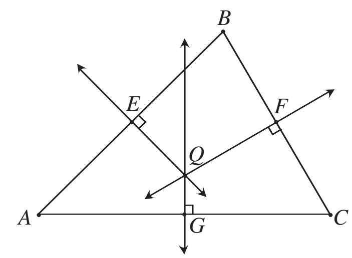
10. The angle bisectors of the angles of △XYZ meet at point Q. XQ = 5 and XP = 4. Find QM. Give a reason for your answer.



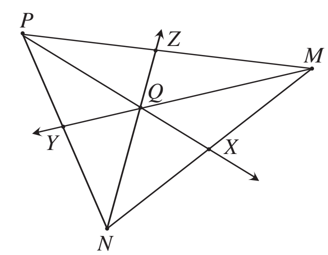
11. The perpendicular bisectors of the sides of △RST meet at point Q. SQ = 11 and QM = 4. Find RQ. Give a reason for your answer.



12. △ABC is given. Perpendicular bisectors of the sides, QE, QF and QG are shown. Can you conclude that QE = QG? If not, explain, and state a correct conclusion that can be deduced from the diagram.

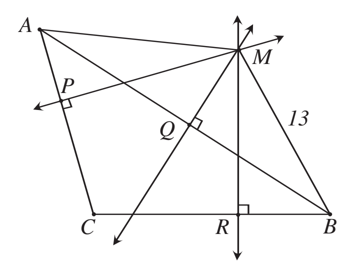


13. Traingle PMN is given. Angle bisectors PX, MY, and NZ are shown. Can you conclude that QZ = QX? If not, explain, and state a correct conclusion that can be deduces from the diagram.



14. The three perpendicular bisectors of the sides of a triangle are concurrent in a point which can be inside the triangle, on the triangle, or outside the triangle. Sketch an obtuse triangle, an acute triangle, and a right triangle showing the perpendicular bisectors of the sides in each to verify each relationship.

15. Triangle ABC is an obtuse triangle. Perpendicular bisectors of the sides meet at point M. MP = 12 and MB = 13. Find AC.



In exercises 16 through 20, complete the statement using always, sometimes, or never.

16. A perpendicular bisector of a side of a triangle __________ passes through the midpoint of a side of the triangle.

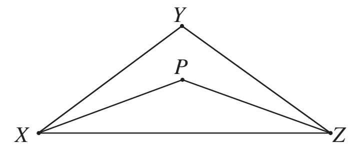
17. The angle bisectors of the angle of a triangle __________ intersect at a single point.

18. The angle bisectors of the angle of a triangle __________ meet at a point outside the triangle.

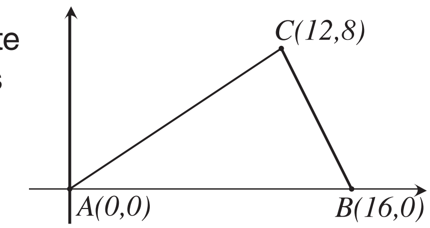
19. The perpendicular bisectors of the sides of a triangle meet at a point which __________ lies outside the triangle..

20. The midpoint of the hypotenuse of a right triangle is __________ equidistant from all vertices of the triangle..

21. $\overline{XP}$ and $\overline{ZP}$ are angle bisectors of $\angle X$ and $\angle Z$ in $\triangle XYZ$. $m\angle XYZ = 112$. Find $m\angle XPZ$.



Use the graph of $\triangle ABC$ and exercises 22 through 24 to illustrate Corollary 83a, about the concurrency of perpendicular bisectors of the sides of a triangle.



22. Find the midpoint of each side of $\triangle ABC$. Use the midpoints to find the equations of the perpendicular bisectors of the sides of $\triangle ABC$.

23. Using your equations from exercise 22, find the intersection of two of the lines.

24. Show that the point in exercise 23 is equidistant from the vertices of $\triangle ABC$.

Unit VI — Circles

Part D — Circles and Concurrency

Lesson 2 — *Theorem 84 - "If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic."*

Objective: To investigate a relationship between chords and arcs of a circle, and to prove these theorems directly, using previously accepted definitions, postulates and theorems.

Important Terms:

Concurrent – From two Latin words meaning "to run together", this term deals primarily with intersecting lines, or line segments. Formally, if two or more lines contain the same point, they are said to be concurrent. (Note: In Geometry, because it is more significant when three or more lines contain the same point, the formal geometric definition of concurrency relates to three or more lines.)

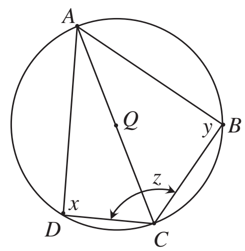
Concyclic – From two Latin words meaning "together on a circle", this term deals primarily with points on a circle. Formally, if two or more points lie on the same circle, they are said to be concyclic. (Note: In Geometry, because it is more significant when three of more points lie on the same circle, the formal geometric definition of concyclic points relates to three or more points.)

Cyclic – A specific term which refers to a relationship between polygons and circles. Formally, when there is a circle which contains all of the vertices of a polygon, then that polygon is cyclic.

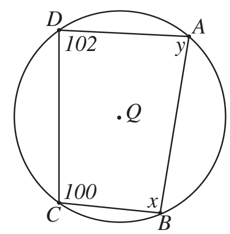
Lesson 2 — Exercises:

1. Prove Theorem 84 - "If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic." (Converse of Corollary 67b)

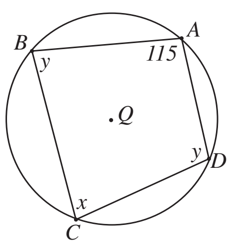
2. Find the values of *x, y,* and *z*.
mBCD = 136°



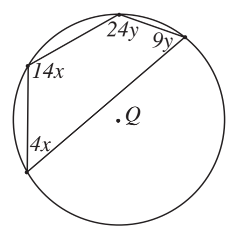
3. Find the values of *x, y,* and *z*.
mBCD = z



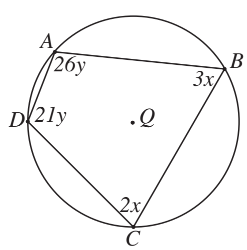
4. Find the values of *x, y,* and *z*.
mABC = z



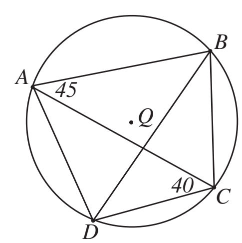
5. Find the values of *x* and *y*.



6. Find the values of *x* and *y*.



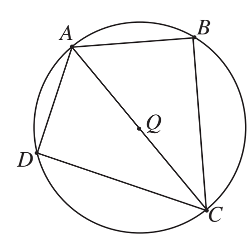
7. Find the measure of the angles of quadrilateral ABCD.



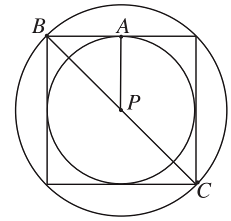
8. Prove that trapezoid inscribed in a circle, is an isosceles trapezoid.

9. Suppose that ABCD is a quadrilateral inscribed in a circle, and that AC is a diameter of the circle.

If m∠A is three times m∠C, what are the measure of all four angles?



10. Show that the ratio of the radius of the inscribed circle to the radius of the circle circumscribed about a square is   to 2.

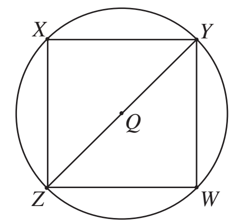


11. All regular simple polygons are cyclic. A circle contains 360 degrees. How many degrees are in each arc of a circle circumscribed about:

a) An equilateral triangle?
b) A square?
c) A regular hexagon?
d) A reular octagon?

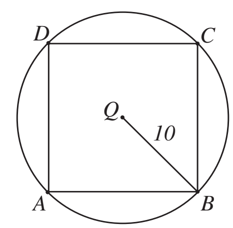
12. Given: Quadrilaterl XYWZ is cyclic.
$\overline{ZY}$ is a diameter of circle Q.
$\overline{XY} \parallel \overline{ZW}$

Prove: $\widehat{XY} \cong \widehat{ZW}$



13. Square ABCD is cyclic

a) Find m∠ADB
b) Find AB
c) Find the distance from point Q to $\overline{AB}$.

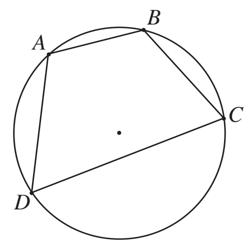


In exercises 14 through 19, tell if the given quadrilateral can always be inscribed in a circle. Explain each answer.

14. Square
15. Rectangle
16. Parallelogram
17. Kite
18. Rhombus
19. Isosceles Trapezoid

20. The line segment in a quadrilateral drawn from a midpoint of a side perpendicular to the opposite side is called a maltitude. In a cyclic quadrilateral, maltitudes are concurrent. Using a straightedge, sketch a cyclic quadrilateral and its maltitudes and verify this relationship.

21. Theorem 84 tells us that the opposite angles of a cyclic quadrilateral are supplementary. There is also a theorem known as Ptolemy's Theorem which tells us the relationship of the lengths of the segments determined by the vertices of a cyclic quadrilateral.



In the given diagram, quadrilateral ABCD is cyclic. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are the sides of the quadrilateral. $\overline{AC}$ and $\overline{BD}$ are the diagonals of the quadrilateral. The sum of the products of the measure of the two pairs of opposite sides is equal to the product of the diagonals.

$$AB \cdot DC + BC \cdot AD = AC \cdot BD$$

Draw a rectangle and apply Ptolemy's Theorem to derive the Pythagorean Theorem.